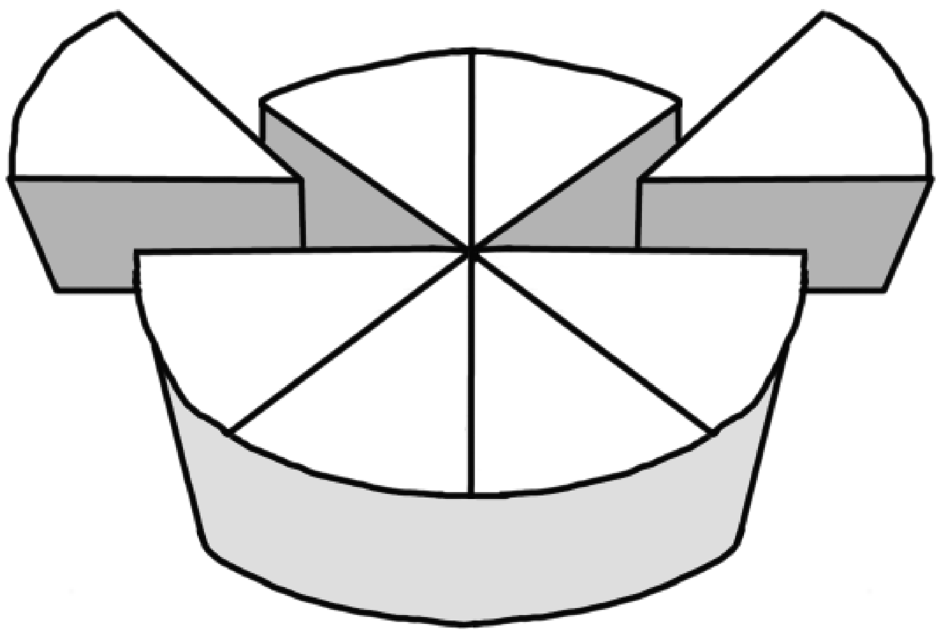
FRACTIONS
DECIMAL FRACTIONS
PERCENT



A VISUAL MEANING-BASED APPROACH TO LEARNING

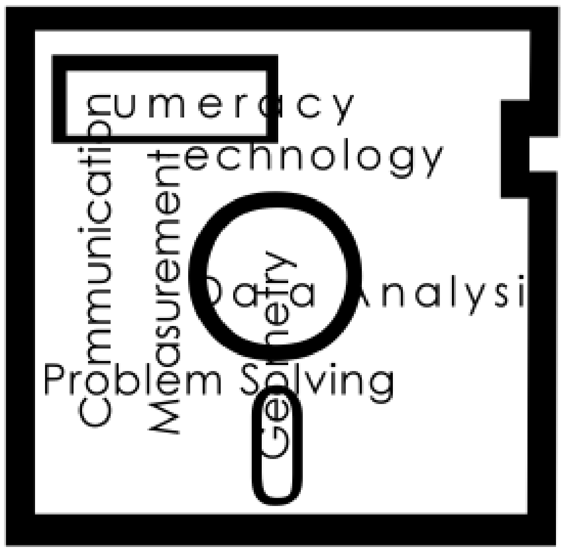
Teaching Notes

Teaching Fractions with Meaning

There are many sources and avenues for teaching fractions with meaning. Be sure to explore these sources fully before you embark on utilizing the following processes for teaching and understanding fractions. Problems surrounding the weekly graph, sports activities, and cooking recipes will provide you with many meaningful sources of study that involve fractions. Connections may eventually be made to the multiplication and division of whole numbers and to decimal fractions. The sooner you tackle these concepts in the school year, the sooner you may begin to build strong number sense in your students.

Research informs us that children demonstrate most effective recall when they are encouraged to visualize. Our visual memories are often very clear and easily retrievable. Memory is also enhanced by personal experience. Students therefore need to create personal meaning when studying fractions. Little league scores, personal batting averages, best free-throw averages and in-class data collection (such as favorite pizza, movie, TV show, or sports heroes) will all help a child to make personal meaning as he reaches for an understanding of fractions.

Movement is a central part of the memory-making process and must be part of every new lesson as fractions are introduced in the classroom. Whenever children are creating fractions, have them perform folding, coloring and cutting activities and even, where appropriate, some role-playing activities. Students need to work from the concrete (folding and role-play) to the representational (pictures and number lines), before finally proceeding to the abstract (numerals).



Strategies described in this chapter will prove useful whenever you are demonstrating the essential role that fractions and decimal fractions play in our daily lives. Although the concepts presented in this chapter could be taught in a few weeks, you will find it more profitable to present different ways of understanding fractions over a period of time, in order to allow students time to integrate their understandings visually. This approach also reduces both teacher and student stress, as it allows all participants a lengthier learning period.

You may well find the first time that you teach children the tenths or hundredths connection to building decimal fractions or percentages a little chaotic, if not confused. However, persistence with this activity will prove profitable eventually, especially where students are not in any way pressured to acquire concepts rapidly, in order, perhaps to move ahead to the next page or the next chapter of the text. Where teachers maintain a “mental” time-line goal that aims for student mastery by the end of the year, the steady growth in student understanding of the connections between common fractions, decimal fractions and percentages will often prove remarkable as the year unfolds.

Current brain research informs us that **we learn most effectively where the learning is presented in "bigger chunks" over longer periods of time.** The advantage to this approach is that instructors do not expect students to understand complex decimal fractions immediately. Students are therefore provided with deeper understanding and meaning-making opportunities, as they are exposed to learning over a longer period of time. Teacher expectations remain low initially, so that the threat of failing is minimized and the conditions for understanding decimal fractions at a very complex level are enhanced and extended. This process may be supported by evaluation procedures that involve the use of portfolios and journals, as these procedures promote processing and reflective thought in young learners.

a Possible "One Day a Week" Plan for the Year

September:
- Introduce the weekly graph.
- Make the connections to the Canadian Football League.
- Complete the whole school survey.
- Perform measurement activities (m, dm, cm, mm).
- Include the measurement of student heights.
- Ensure that students calculate their test scores in percentages. *(grade 5)*

October:
- Students now assume responsibility for planning the weekly graph.
- Relate to cooking at Thanksgiving (emphasize ratios).
- Make the connection to the National Football League.
- Make the connection to the National Hockey League.
- Analyze one question from the whole school survey.

November:
- Continue the weekly graph, ensuring that everyone has the opportunity to plan, or initiate, graphing activities.
- Introduce the National Basketball League.

December:
- Connect activities to holiday shopping and related data-analysis studies.
- Relate activities to holiday cooking (emphasize ratios).
- During this month, take a brief break from weekly graphing activities.

January:
- Use the study of area as a source for generating fractions.
- In basketball, relate the number of successful basketball free throws to parts of each possible total: for example, fifths, tenths and twentieths.
- Resume the weekly graph.
- Include the spelling activity found in the Data Analysis chapter.

February:
- Continue with the weekly graph.
- Analyze one question from the whole school survey.
- Include the "Win a Free Pop" activity found in the Data Analysis chapter.

March:
- Use volume and capacity as a source of fractions (including thousandths).
- Analyze one question from the whole school survey.
- Include the "Card Sums" activity found in the Data Analysis chapter.

April:
- Connect activities to the start of baseball season.
- Connect activities to problems related to cooking.
- Analyze one question from the whole-school survey.
- Include earthquake-simulation activities found in the Data Analysis chapter.

May:
- Use Little League scores (team wins, personal batting averages and fielding averages.
- Relate activities to whole-school track and field scores. Analyze one question from the whole-school survey.

June:
- Relate activities to cooking (emphasize ratios).
- Relate activities to year-end trips.

Note: The activities found above will appear in a different order in the Yearly Plans. The order of activities depends upon specific grade level.

It is important that you constantly look for fractions in the home and school environment, in order to make the teaching of fractions more meaningful for students. Sample meaningful sources of fractions include the following:

- ✓ The fraction of questions answered correctly on an in-class test.
- ✓ The fraction of students who have returned a form to school on time.
- ✓ The percentage of girls born in January.
- ✓ The fraction of students in the whole class who like mustard on hot dogs.

You will find that the possibilities are endless once you start looking. **Try to introduce at least one new meaningful fraction each week.** You will no doubt be astounded to see the interest each new example generates. As you develop an understanding of fractions, always ensure that students are able to relate any fraction to zero, one-half, and one. For example, ask the following questions:

- ✓ Is the fraction of students who returned this week's form closer to **0**, **1/2**, or **1**?
- ✓ Does the number of girls in the class represent more or less than **50**% of the whole class?

Teaching Fractions as Parts of a Whole

- Distribute either hundredths squares or ordinary rectangular paper. Rectangles are easily created by folding a piece of paper into quarters and cutting it into four rectangles **This is an excellent way to recycle paper that has been used on one side.**

- It is important that wherever possible you **relate understanding to students' real life experience.** As most students have an innate sense of one half, initially begin by teaching an understanding of fractions with discussion of one half of a whole. You will later be able to use this understanding as a reference point for the teaching of all fractions, and you will be able to make the connection to **50**%.

- Have each student fold a piece of paper into half, unfold the paper, and then mark the fold line with a felt pen. Ask: **How many parts are there?** (There are two parts.) Ask: **What is each part called?** (Each part is called one half.) Instruct students to coluor one half.

- Have each student refold the paper in half once again. Explain to students that where they use a horizontal fold in step one, they must use a vertical fold in step two, and that they must continue to create alternate folds in this way. Students then open their folded papers and mark the new fold lines with felt pens. Ask: **How many parts are there in all?** (You will hear fourths, one-fourth and one quarter. Encourage all responses.) Ask: **What is the name of the coloured fraction?** (The coloured fraction is called two fourths.)

- Have students refold their papers in half once again. Students will now have created eighths. Encourage students to note the relationship between: $\frac{1}{2}$ $\frac{2}{4}$ $\frac{4}{8}$

Ask: **Can you find the next two terms $\frac{8}{16}$, and $\frac{16}{32}$ by continuing the folding process?**

Ask: **Can you find the relationship between $\frac{1}{2}$ and each of the fractions you have discovered as you folded your paper?**

Explain to students that $\frac{1}{2}$ becomes $\frac{2}{4}$ when the numerator and denominator are multiplied by $\frac{2}{2}$, as $\frac{2}{2}$ equals one. This is why the value of the fraction does not change, even though the form of the fraction changes.

Now ask: **What fraction in the pattern do you think would occur, or "fit in", between $\frac{2}{4}$ and $\frac{4}{8}$?** (This fraction would be $\frac{3}{6}$.)

- For example, one way to utilize prime factors is as follows: $\frac{2}{4} = \frac{2}{2 \times 2} = \frac{1}{2}$

  or

  $$\frac{4}{8} = \frac{2 \times 2}{2 \times 2 \times 2} = \frac{1}{2}$$

- This fraction reduces in this way because $\frac{2}{2}$ equals **1**. **It is essential that you emphasize the property of one, and that you avoid using the terms "cancel" or "cancel out".** The term "cancel" has no real contextual meaning for students, and it may even leave some students with the impression that in mathematics one may occasionally "cancel", or remove items, in certain situations. Students have difficulty remembering which situations permit cancellations, or they may tend to over-generalize the process and to apply it to other situations where the process is not applicable.

- Connect the fractions you have studied to the **Visual Percentage Calculator.** *(See Appendix.)* This tool relates equivalent fractions to area and to the number line.

Teaching Fractions as Part of a Number Line

- **It is essential that you present fractions in a variety of meaningful contexts.** Students will then be able to make personal meaning of the fractions they encounter and to relate that understanding to their own unique experiences. Students do not always encounter fractions in the contexts that adults commonly do, and so **it is important that an introduction to fractions is presented with a variety of meaningful contexts.** However, we often resist exposing students to a variety of contexts as we falsely assume that this practice will serve only to confuse students. The unfortunate result is that students are only able to apply their understanding of fractions within the very narrow context of classroom studies.

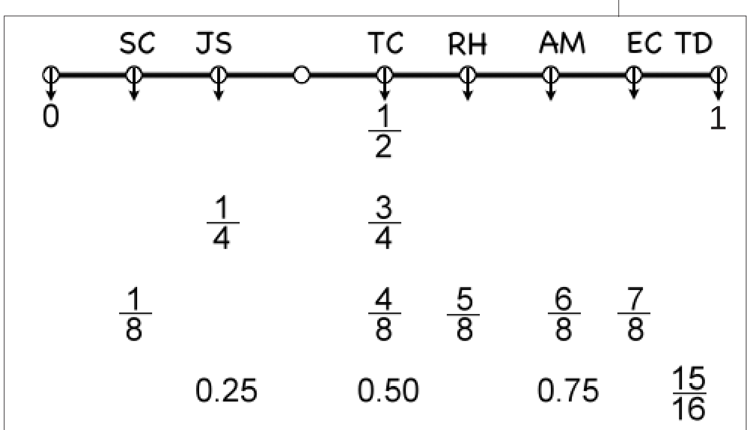
- Have students fold recycled paper into fractional halves, quarters or eighths as discussed above and then to colour in one specific fraction. Emphasize that the selection of fraction for colouring is entirely one of personal choice, but that each student must select a fraction that is different from his neighbor's fraction. It is often useful to have students perform this activity on the reverse side of hundredth-squared paper, so that they will eventually be able relate their findings to decimal fractions and percentages.

- Ask each student to draw a number line in his or her math workbook and to label one end of the line **0** and the other end **1**. Draw a number line **2**m in length on the chalkboard and label the ends of the line **0** and **1**.
Ask: **"Does anyone have a coloured fraction that could be placed exactly half way along the number line?"** Hopefully, someone in the class will have colored in a fraction equivalent to **1/2**, **2/4** or **4/8**. Label the chalkboard number line with halves, quarters and eighths. Wherever a student suggests a fraction, place that student's initials above the number line where his fraction occurs. At the end of the process you may want to ask each student thus identified to stand in front of his initialed place on the number line.

You may well be fortunate on many occasions to have three different students suggesting one half, two fourths and four eighths. Follow by asking if anyone has coloured a fraction that stands between **4/8** and **1**. Ask students to come forward and to show "about where" this fraction would appear on the number line. As students come forward, place their initials above the correct number-line placement. Students eventually stand directly in front of their initials. Continue with this activity until as many parts of the number line show fractional placements as possible. Some classes are able to maintain this activity for longer periods than others before becoming bored with the repetition. This process works effectively where questions such as those that follow are employed. **Note: The possibilities are endless.**

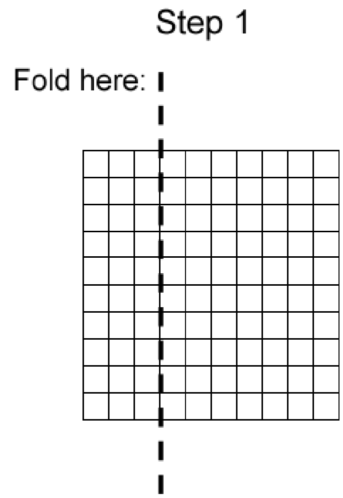
- Does anyone have a fraction greater than Trevor’s and less than **1**?
- Does anyone have a fraction that would “fit” between Trevor’s fraction and Amy’s fraction?
- Does anyone have fraction equivalent to Amy’s fraction?
- Does anyone have a fraction between **0** and Melissa’s fraction?

- Follow by having the class place decimal fractions along the number line. Start with $\frac{1}{2}$ and prove that this fraction is equivalent to **0.50** by having each student who coloured in colour an equivalent portion on the reverse side of his or her paper. This activity is particularly effective where students are using paper with hundred squares on the reverse side, equivalent decimal fractions are then easily perceived. Follow by having students find equivalent values using a calculator. Initially have students try $\frac{2}{4}$ and $\frac{4}{8}$. Follow this with $\frac{1}{4}$ and $\frac{3}{4}$, and ask students to try to prove that these two fractions are equivalent to **0.25** and **0.75**.

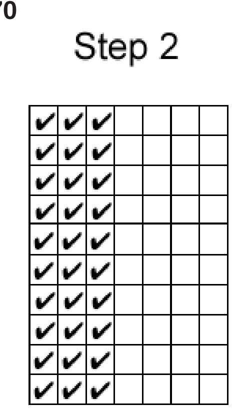


For students in grade five or above, include sixteenths in the process and eventually extend the process to fractions greater than one.

- Finally, ask students to try to represent $\frac{3}{7}$ on hundredth-squared paper. This activity can prove very difficult for some students. Instruct students to fold their papers so that only **70** squares remain visible, as **30** squares are folded under and remain hidden.

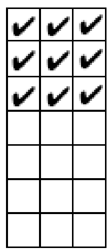


Students then colour three out of every seven squares (or three-sevenths of each row) yellow. Eventually students will have coloured **30** squares in all, or $\frac{30}{70}$.



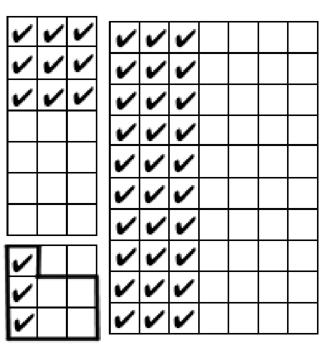
At this point ask students to turn over their papers and look at the **30** remaining squares. Each student then folds under three rows, thus leaving **21** squares, or three columns of seven still visible. Students then colour three sevenths of each column, or nine squares in all.

Step 3



Each student is now looking at nine remaining squares. A group of seven squares is then outlined in felt pen and subsequently three of the seven squares are coloured. Thus, students have coloured **3:7**. Then ask students to unfold their papers and to count the number of colored squares. This representation should be as follows:
30 + **9** + **3** = **42** out of **100**.

Step 4



- Reinforce the connections that students are now able to perceive **by showing three sevenths as a decimal on a calculator.**
  This operation will yield:
  0.428571428571428571428571.
  Most calculators only show **0.4285714**, and so students do not observe that this is a repeating decimal fraction. This operation presents an interesting topic for in-depth further study:
  1. **What fractions yield terminating decimals?** (Halves, quarters, eighths, tenths, sixteenths, twentieths, or any fractions with denominators that have prime factors of only twos and fives.) **Avoid telling, or presenting, this rule.** Instead, encourage students to develop this rule independently. *(See the Decimals, Fractions, and Percent Project.)*
  2. **Which fractions yield repeating decimals?** (Any fraction that in its simplest form has a denominator that has some prime factors other than two or five.) Again, avoid telling, or presenting, the rule.

- Have students enter all fractions and their decimal equivalents on the ***Memorable Fractions*** sheet. You will find a copy of this sheet in the ***Appendix*** of this manual.

- Ensure that students connect some fractions to the ***Visual Percentage Calculator,*** which is also found in the ***Appendix*** of this manual.

- Finally, encourage students to try a few further fractions that they may have been working with and to enter these fractions on their ***Memorable Fractions*** sheets.

Teaching Fractions as Percentages

- Using hundredths squares, emphasize that percent means "out of one hundred". Make the connection with the percent symbol, and point out that this symbol shows a slanted "**1**" with a zero placed at each side of the slant. Refer often to the root word ***cent***, which constitutes the root of ***century***, ***cent***, and ***centimeter***.

- Students should now be able to place **12.5**%, **25**%, **37.5**%, **50**%, **62.5**%, **75**%, **87.5**% and **100**% along a number line.

- Have students place these values on the ***Memorable Fractions*** sheet. *(See the Appendix of this manual for a copy of this sheet.)*

- Teach the relationship of fractions to ratio by reviewing the number line developed above and referring initially to $\frac{3}{4}$. Ask three girls and one boy to stand at the front of the class. Emphasize with your students that three of the four people standing before the class are girls and develop the notation **3:4**.

  Ask:
  - ✓ What fraction of the people standing at the front of the class is made up of girls?
  - ✓ What fraction of the people standing at the front of the class is made up of boys?
  - ✓ What percent of the people at the front are girls? Are boys?

- Follow by asking three more girls and one more boy to come to the front. Have each new addition stand directly in front of a person of similar gender, so that two rows now stand before the class, each with three girls and one boy.

  G G G B
  G G G B

Ask:
- ✓ What fraction of the people in the *back* row is made up of girls? ($\frac{3}{4}$)
- ✓ What fraction of the people in the *front* row is made up of girls? ($\frac{3}{4}$)
- ✓ What fraction of all the people in *both* rows is made up of girls? ($\frac{6}{8}$)
- ✓ Repeat these questions, asking for percentages instead of fractions, and instruct students to check the percentage values on a calculator.

- Follow by asking three more girls and one more boy to come to the front. Have each new addition stand directly in front of a person of similar gender, so that three rows now stand before the class, each with three girls and one boy. Repeat the questions listed above.

- Persist with this activity until you suspect that students may be beginning to lose interest.

- Examine the fractions developed so far with your students and encourage students to look for an inherent pattern. Persist until independent understanding occurs.

$\frac{3}{4}$ $\frac{6}{8}$ $\frac{9}{12}$ $\frac{12}{16}$ $\frac{15}{20}$ ***Note** that all of these fractions are equivalent to 0.75 or 75%.*

$\frac{1}{4}$ $\frac{2}{8}$ $\frac{3}{12}$ $\frac{4}{16}$ $\frac{5}{20}$ ***Note** that all of these fractions are equivalent to 0.25 or 25%.*

Show how some of these fractions are equivalent using the “Power of One” process.

$\frac{3}{4} \times \frac{5}{5} = \frac{15}{20}$ or $\frac{5}{20} \div \frac{5}{5} = \frac{1}{4}$

- Follow by repeating the process described above using fifths. **Do not initially use hundredths squares.** Instead, use tenths squares and then later create a connection to hundredth squares when you ask students to calculate equivalent percentages.

- You may find it effective to focus at some point on twentieths, as these fractions relate well to spelling test scores and can therefore be employed as a meaningful context for the study of percentages and decimal fractions.

- Always follow up by connecting these fractions to the ***Visual Percentage Calculator***.

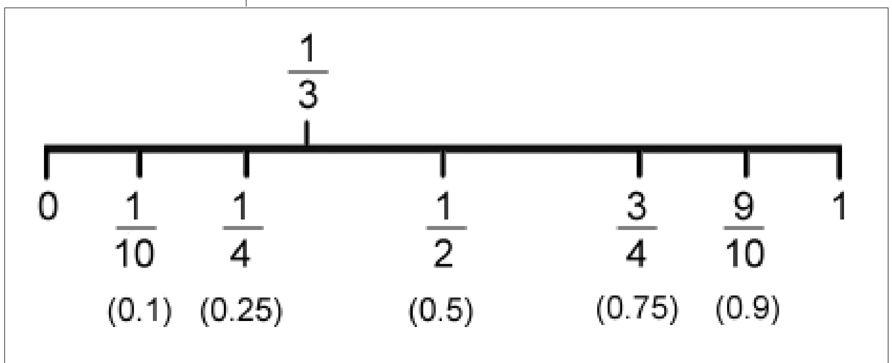
Teaching Fractions with Non-Terminating or Repeating Decimals

- **Begin by presenting a ratio involving thirds to your students.** Then ask each student to fold a hundredth square into equal thirds. This task is most easily accomplished by rolling a hundredth square into a tube and then pressing the tube flat to create three equal thirds. Students will also find this task a little easier where sheets show enlarged squares.

- Instruct students to colour in one third of the reverse side of each sheet and then to try colouring an equal third on the hundredths side. You may wish to have students employ the procedure modeled above for colouring sevenths. Students will discover, however, that they are unable to colour a whole number of squares. One extra square always remains. *(See the **Decimals, Fractions and Percent Project** when showing a relationship to prime factors, remainders, and long division.)*

- Have students try this operation on a calculator. The sequence **0.33333333333333** is unending, and its final yield depends only on the capacity of an individual calculator. Therefore, we employ a rounding process and we round this decimal fraction to **0.33** or **33**%. Later you will find it useful to create meaningful contexts by connecting this decimal to sports event scores (percentage of wins and batting averages, etc.) where scores are recorded as **0.333**.

- Follow by relating the folded model created above (showing **1/3**) to a number line. When relating decimal fractions to a number line, always emphasize each fraction's relationship to **0**, **1/2**, or **1**. As student understanding becomes more sophisticated, relate each fraction's placement along a number line to **0**, **1/10**, **1/4**, **1/2**, **3/4**, **9/10**, and **1**.



- Have students fold thirds into half, so that they thus create sixths. Relate this activity to the number line, noting the relationship between **3/6** and **1/2**, **2/6** and **1/3**, **4/6** and **2/3**. Follow by having students create ninths and twelfths and then to relating their understanding of these fractions to ratios and percentages, using numbers lines and coloured hundredth squares. Wherever possible reinforce that equivalent fractions can be created using the **"Power of One"** process.

- Where students become confident with this practice, **introduce fractions that are greater than one**, and place these fractions along a number line, also. This operation will prove particularly useful when teaching long division, as it provides a meaningful understanding of remainders.

- **"Power of One Process"**
  To create an equivalent fraction use a form of one (**2/2**, **3/3**, **4/4**, **5/5**....) and multiply.

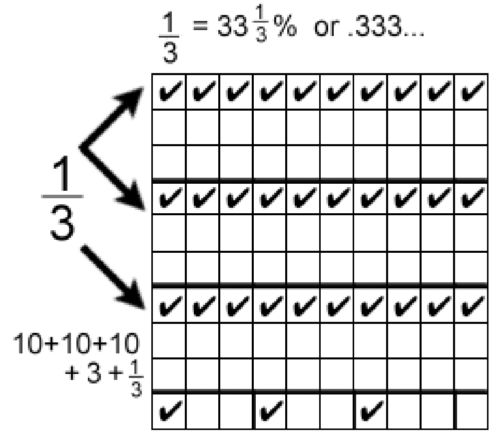
$$\frac{1}{3} \times \frac{4}{4} = \frac{4}{12}$$

  To create the **simplest equivalent fraction** write the prime factorization of the numerator and denominator and find the ones.

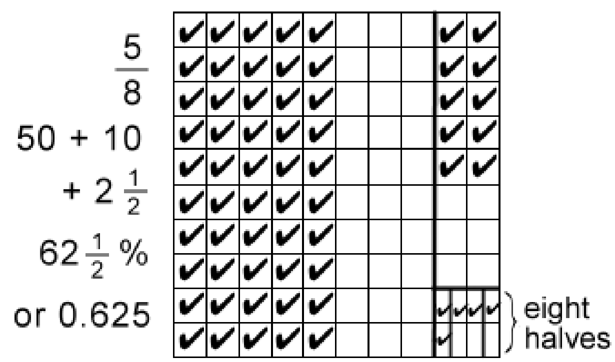
$$\frac{20}{30} = \frac{\cancel{2} \times 2 \times \cancel{5}}{\cancel{2} \times \cancel{5} \times 3} = \frac{2}{3}$$

TEACHING FRACTIONS, DECIMALS AND PERCENT USING HUNDREDTHS SQUARES

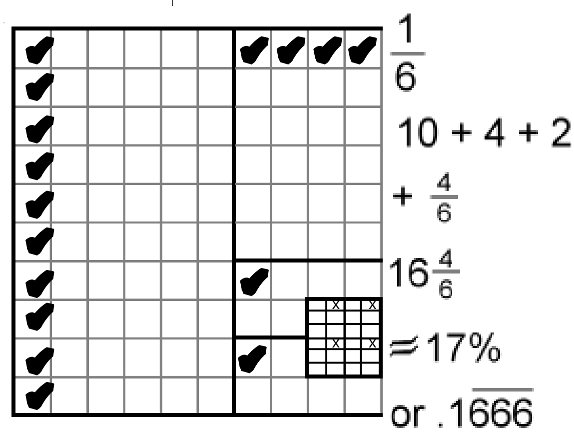
- Begin by deciding whether you wish students to fold or outline specific groups of squares on their hundredth sheets. Classroom experience tends to show that **outlining with felt pens proves easier for students than folding does.** For example, where students are asked to create a visual model showing **1/3**, they first outline **30** squares, and then they colour one out of each three squares, or **10** out of **30** squares. Students repeat this process for two additional groups of **30** squares. Thus, **10** squares still remain untreated on each hundredth sheet. These **10** squares are separated into three groups of three and outlined as such. In each of these three groups, one out of every three squares is coloured. Thus, one odd square still remains uncolored. The visible percentage model created in this way is seen as approximately **33**%. Students who are able to work with greater precision should shade in **1/3** of this final remaining square, yielding **33 1/3**% or **0.333** returning. Students are often fascinated with this endlessly repeating process. Where shading is accomplished on thousandth-squared paper, one odd square will still remain. This operation may be successfully shown on an overhead projector using a thousandth-squared transparency. To show millionths, you would need to divide each part of a thousandths square into **1000** parts (which is almost impossible to demonstrate). This operation demonstrates clearly for students that as decimals appear to become ***larger***, the corresponding parts become ***smaller***.



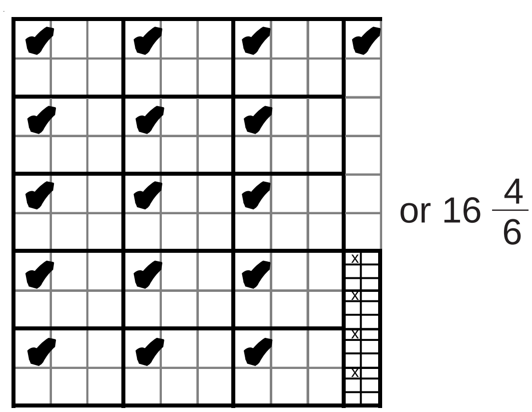
- In order to create a visual showing **5/8**, students first outline **50** squares and then colour five out of every eight squares, or **50** out of every **80** squares. This process is continued for the remaining **20** squares, as students colour **10** of **16** squares. Thus, four untreated squares remain. Students divide each of the four squares in half and then they proceed to colour in five of these eight halves. This operation yields **2 1/2** squares for a total of **62 1/2**. Therefore, the percentage coloured is seen as exactly **62.5**%, or **0.625**.



- When creating a visual showing **1/6**, students first outline **60** squares and then they colour one out of every six, or **10** out of every **60** squares. This process is repeated for the remaining **24** squares, as students colour four out of every **24** squares. Thus, **16** squares still remain untreated. Students then create two sets of six squares and color one out of every six in each set. Each of the four remaining squares is divided into sixths, and students then color in one out every six, yielding **4/6**. This operation yields a total of **16 4/6**, or approximately **17**%, or **0.1666**….



- Some students independently discover that it is easy to just find groups of six. They seem to prefer a **2** x **3** array rather than a **1** x **6** as shown above. Using this method the array would look like:



Relating Fractions and Decimals to Measurement

Select at least five items and measure the length of each. Begin by measuring with a piece of adding machine tape that measures one metre. Then mark the tape in tenths and measure in metres and decimetres. Next measure with a metre stick in metres and centimeters. Finally measure in metres and millimeters. As you complete each operation, initially express each measurement as a mixed number and then as a decimal fraction.

Object	m	dm	Fraction and Decimal	cm	Fraction and Decimal	mm	Fraction and Decimal
Example: Trevor's Height	2	17	$1\frac{7}{10}$ or 1.7	168	$1\frac{68}{100}$ or 1.68	1683	$1\frac{683}{1000}$ or 1.683
1.							
2.							
3.							
4.							
5.							

An example has been written in the table above and another example follows:

If your height measures **16 dm**, this measurement may be expressed as $\mathbf{1\frac{6}{10}}$ in fractional notation and as **1.6 m** in decimal notation. If you measure your height in **cm**, this measurement may be expressed as **164 cm** or $\mathbf{1\frac{64}{100}}$ = **1.64 m**. (This fraction may also be expressed as $\mathbf{\frac{164}{100}} = \mathbf{\frac{41}{25}} = \mathbf{1\frac{16}{25}}$.) If you choose to continue and to measure your height in **mm**, this measurement may be expressed as **1639 mm** or $\mathbf{\frac{1639}{1000}} = \mathbf{1\frac{639}{1000}}$ = **1.639**.

Note: **The first fraction shown here is in *improper* form, the second fraction in *mixed* form, and the latter fraction is in *decimal* fraction form. Instruct students to write fractions in more than one form, as this practice often aids students to gain understanding of subsequent steps in a given problem.**

Operations With Fractions

Both **teachers and students in elementary school often view fractions as a challenging area of study**. Students often encounter difficulty understanding fractions, while teachers often fail to present fractions in concrete, meaningful ways and instead employ only "rule and practice" methodologies. "Rule and practice" sometimes creates the misleading impression that students are able to manipulate fractions easily. However, many students learn the rules but then either forget or ignore the related context in which those rules are applied. They then misapply the rules in future encounters with problems involving fractions.

<u>This manual is based on the belief that</u>:

1. Students are less likely to forget rules that they themselves create, as these rules relate to their own unique life experiences.
2. Opportunities should be provided in the classroom for concrete learning experiences.
3. Learning is visual.
4. Mathematical connections should be explored in which fractions are related to a wide variety of learning contexts (sports, cooking, games, decimals, number lines, diagrams).
5. Initial practice with fractions should be meaningful, and students should never be permitted to practice mistakes repeatedly as they work through endless worksheets.
6. Initial practice should be followed by discussion regarding the mathematical thinking employed and errors that have occurred.
7. Concepts related to fractions should not be assigned as homework until the student has fully mastered the relevant concept.

Adding Fractions

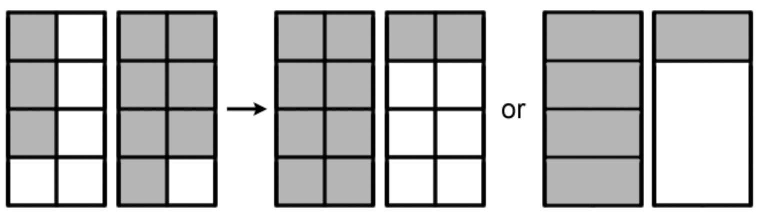
Teachers usually find it effective initially to show the addition of fractions with pictures and number lines. Note that previous sections of this chapter suggest teaching an understanding of equivalent fractions by both folding squared paper and by making connections to the number line. You will find the employment of folded squared paper very useful in problem-solving contexts related to an understanding of fractions.

Example One:

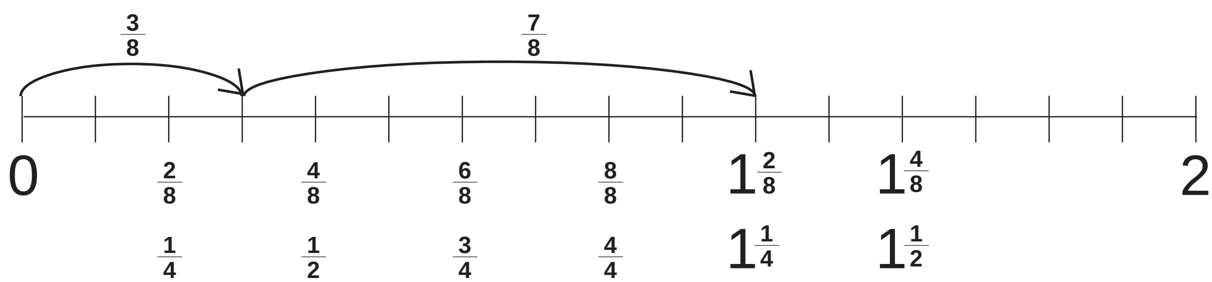
Joanne has $\frac{3}{8}$ of a chocolate bar, and Misty has $\frac{7}{8}$ of a chocolate bar.

How much do they have altogether?

Show your answer using a diagram.



$\frac{3}{8} + \frac{7}{8} = \frac{10}{8}$ or $\frac{5}{4}$ or $1\frac{2}{8}$ or $1\frac{1}{4}$ or **1.25**



Note that in this example each type of diagram should be accepted and encouraged. Occasionally you may wish to ask students to provide both a diagram and a number-line representation of the answer. It is also essential that you accept all three forms of the correct answer (improper fractions, mixed fractions, and both forms in simplified format). Students should receive five marks for the correct answer and bonus marks for including alternative forms of the correct answer. In this case, five marks are awarded for the correct answer ($\frac{10}{8}$), and bonus marks are awarded for including the three alternate forms that show this answer. Where a student provides the decimal format (**1.25**), he or she should receive an additional bonus mark.

Encourage students to create their own questions independently. Where a student has successfully shown a correct diagram for one example, he or she may then proceed to show only the number sentence and alternate answer formats when completing additional examples.

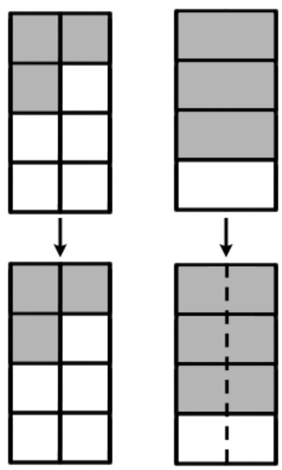
Note that some students may create questions that include fractions with different denominators.

Example Two:

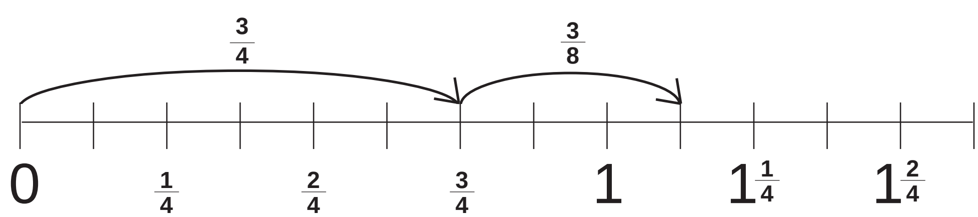
James has $\frac{3}{8}$ of a chocolate bar, and Mark has $\frac{3}{4}$ of a chocolate bar.

How much do they have altogether?

Show your answer using a diagram.



$$\frac{3}{8} + \frac{3}{4} \rightarrow \frac{3}{8} + \frac{6}{8} = \frac{9}{8} \text{ or } 1\tfrac{1}{8} \text{ or } 1.125$$

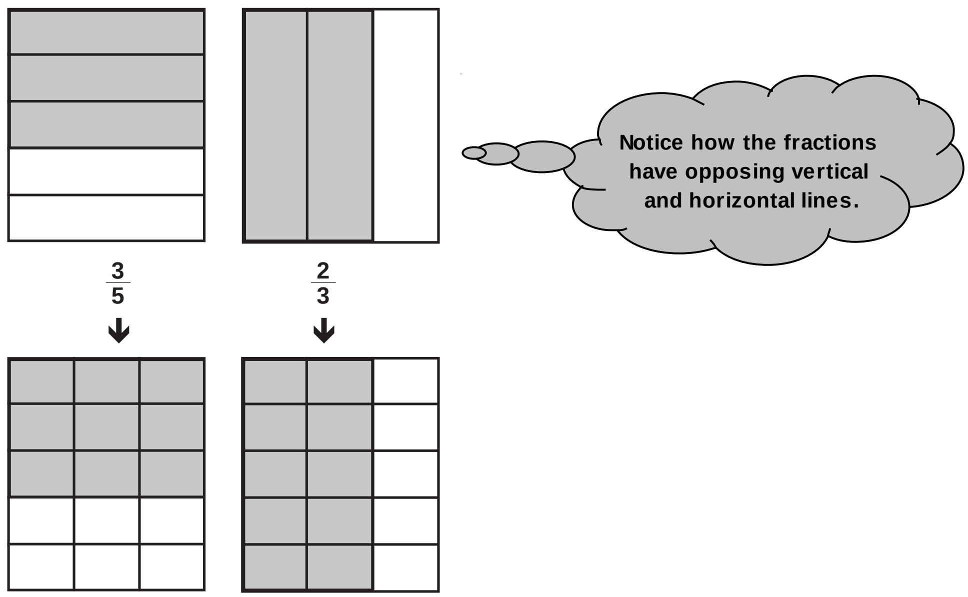


Example Three:

James has $\frac{3}{5}$ of a chocolate bar, and Mark has $\frac{2}{3}$ of a chocolate bar.

How much do they have altogether?

Show your answer using a diagram.



$$\frac{9}{15} + \frac{10}{15} = \frac{19}{15} \text{ or } 1\frac{4}{15}$$

Note that the fractions shown in this diagram cannot be combined where the fractional parts are not congruent. However, the parts may be rendered congruent by cutting the chocolate bars in the same direction. (Three fifths is instead shown on a chocolate bar cut to show nine fifteenths.)

This method of creating congruent fractional parts should be encouraged from the outset of instruction. Begin immediately to ask the question: **"Can we use the *power of one* to make equivalent fractions?"** Hopefully, at least a few of your students will soon understand that equivalent fractions can be created by using the prime factors of the denominators.

In the three previous examples:

Example One:

use the "Power of One"

$$\frac{3}{2 \times 2 \times 2} + \frac{7}{2 \times 2 \times 2} = \frac{10}{2 \times 2 \times 2} \text{ or } \frac{\cancel{2} \times 5}{\cancel{2} \times 2 \times 2} \text{ to get } \frac{5}{4} \text{ or } 1\frac{1}{4}$$

Example Two:

"Power of One"

$$\frac{3}{2 \times 2 \times 2} + \frac{3}{2 \times 2} = \frac{3}{2 \times 2 \times 2} + \frac{3}{2 \times 2} \times \left(\frac{2}{2}\right) = \frac{3}{2 \times 2 \times 2} + \frac{6}{2 \times 2 \times 2} = \frac{9}{8} = 1\frac{1}{8}$$

Example Three:

"Power of One"

$$\frac{3}{5} + \frac{2}{3} = \frac{3}{5}\left(\frac{3}{3}\right) + \frac{2}{3}\left(\frac{5}{5}\right) = \frac{9}{15} + \frac{10}{15} = \frac{19}{15} \text{ or } 1\frac{4}{15}$$

"Power of One"

It is essential that middle-school teachers relate the skills needed when adding fractions to the principles needed for solving algebraic equations. *(See the **Algebraic Connections** chapter of this manual.)*

Equation Solving Principles

- **Magic of Zero:** Often the first step in solving an equation successfully is that of eliminating terms. This is almost always done by creating zeros.
- **Power of One:** Once the equation consists of one term on each side of the equation, the objective then is to make the coefficient of the variable term into a *one*.
- **Balance:** What is done to one side of an equation must be done to the other side of the equation.
- **Equivalence**: When applying the principle of balance, it is possible to use different but equivalent forms of zero or one.

Subtracting Fractions

The subtraction of fractions may be viewed as the reverse process of addition. However, the diagrams that show subtraction are substantially different in appearance than those that show addition.

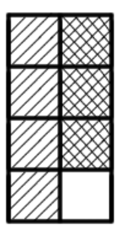
Example One:

Joanne has $\frac{7}{8}$ of a chocolate bar left when her friend Misty arrives.

She gives Misty $\frac{3}{8}$ of her bar.

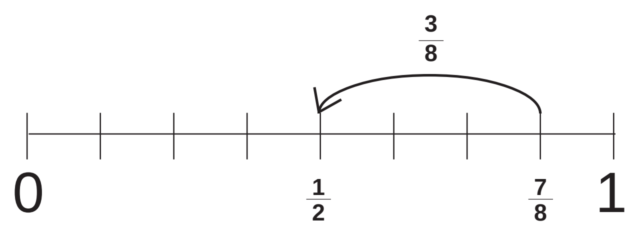
How much of Joanne's chocolate bar still remains?

Show your answer using a diagram.



"Power of One"

$\frac{7}{8} - \frac{3}{8} = \frac{4}{8}$ Use "Power of One" $\frac{\cancel{2 \times 2}}{\cancel{2 \times 2} \times 2} = \frac{1}{2}$ or **0.5**



Award five marks for both successfully completing the diagram and for providing the sentence that shows the answer of $\frac{4}{8}$. Award bonus marks for writing $\frac{1}{2}$ and for providing the decimal version of **0.5**. Awarding bonus marks in this way encourages students to think of alternative ways in which a fractional answer may be presented. Understanding alternative formats for representing fractional numbers is a skill often required when solving problems or when writing multiple-choice tests in which the "lead" answer is presented in equivalent format.

It is essential that as an instructor you make mathematical connections to both *parts* of a whole diagram and to visual number lines. Both diagrams and number lines are useful applications, and either one may make sense to individual learners, depending on each learner's previous experience and learning style.

Multiplying Fractions

Many students will have experienced multiplying fractions in their lives. However, many may not have been shown the mathematical way of writing and recording the multiplication of fractional numbers. **There are two basic concepts that students must grasp:**

1. When a fraction is multiplied by a fraction showing a value of less than one, the answer will always be less.
2. The word "of" means *multiplication* in mathematics. (I have four bags of six candies means I have *four multiplied by six* candies, which equals a total of twenty-four candies).

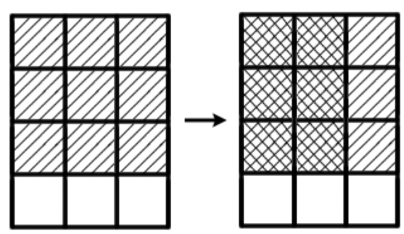
Students who have seen a tenths square on an overhead projector (oriented vertically) placed upon another tenths square (oriented horizontally) will understand that one-tenth of one-tenth equals one-hundredth. Similarly, most students know that one-half of one-half equals one-quarter. In earlier grades, students will have had some experience with creating equivalent fractions when multiplying fractions such as five-fifths and three-thirds.

An understanding of the multiplication of fractional numbers may also be shown on the number line.

$\frac{1}{10} \times \frac{1}{10} = \frac{1}{100}$ $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$ $\frac{3}{5} \times \frac{3}{3} = \frac{9}{15}$ $\frac{2}{3} \times \frac{5}{5} = \frac{10}{15}$

When showing the multiplication of more difficult fractions (such as two-thirds of three-quarters) it is essential to show this operation in a diagram. Hopefully, one of your students will soon announce that the operation is easily accomplished by: "Multiplying the numerators and then the denominators and finding equivalent forms if possible."

$\frac{2}{3} \times \frac{3}{4}$ Draw $\frac{3}{4}$ and then find two-thirds of it. The concept of the lowest common denominator used in adding and subtracting can be useful for making these drawings on graph paper (in this case **12**). Draw $\frac{3}{4}$ using **12** squares.



Now find $\frac{2}{3}$ of the shaded part, using a diffent slant. The result is **6** parts or $\frac{6}{12}$ of the original rectangle.

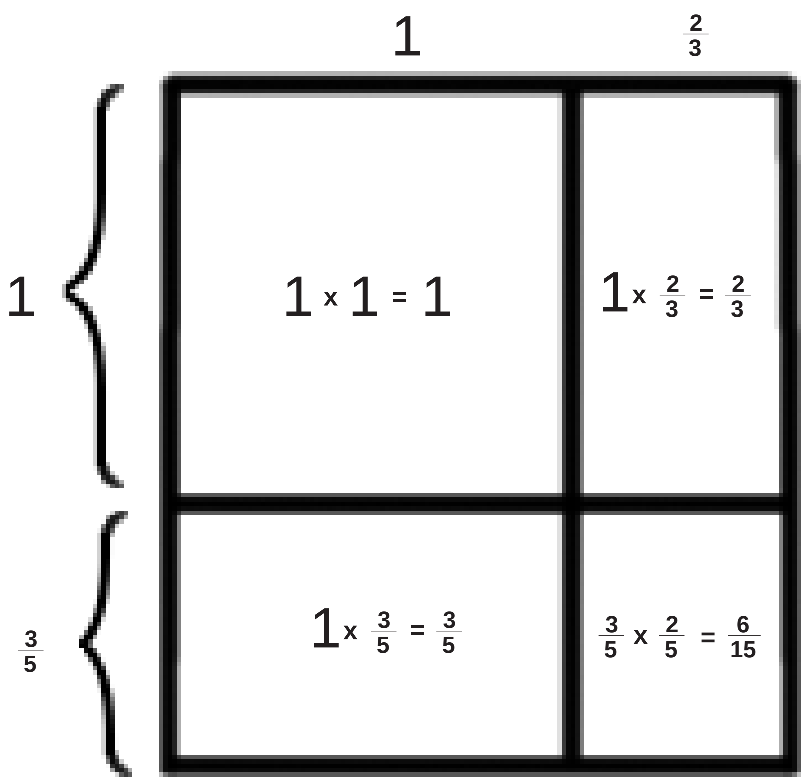
$\frac{2}{3} \times \frac{3}{4} = \frac{6}{12}$ or $\frac{\cancel{2} \times \cancel{3}}{2 \times \cancel{2} \times \cancel{3}} = \frac{1}{2}$ or **0.5**

"Power of One"

It is probably advisable to create the drawings along with your students before showing your students the multiplication format. Eventually students will need to know how to multiply mixed numerals, and this operation will prove easier if the distributive form of multiplying whole numbers has been employed. This format has many useful applications later in factoring and expanding polynomial expressions.

It is essential that instructors understand that **the most important skill students need when multiplying fractions is the ability to understand fractions in meaningful contexts**. The ability to manipulate fractions rapidly with the application of rote rules is a far less useful skill. The ability to apply rote rules after continued meaningless practice will prove ineffective when students are required to think and to understand why the rules are inapplicable in other, diverse situations. <u>There is only one important rule a student needs to remember:</u>

Always ask:
"Does my answer make sense?"



Principal of *one* used to get:

$1 \rightarrow \frac{15}{15}$

$+ \frac{2}{3} \rightarrow \frac{10}{15}$

$+ \frac{3}{5} \rightarrow \frac{9}{15}$

$+ \frac{6}{15} \rightarrow \frac{6}{15}$

$\frac{40}{15}$

$$1\frac{3}{5} \times 1\frac{2}{3} = \left(1 + \frac{3}{5}\right)\left(1 + \frac{2}{3}\right)$$

$$\frac{8}{5} \times \frac{5}{3} = \frac{40}{15} \longrightarrow \text{or } 2\frac{2}{3} \longleftarrow \text{or } \frac{8}{3} \longleftarrow \frac{40}{15}$$

Note: Your students will find it easier to understand the multiplication of fractions where you have consistently read decimal notations as fractional numbers. <u>For example</u>:

$$1\tfrac{3}{10} \times 1\tfrac{2}{10} = \tfrac{13}{10} \times \tfrac{12}{10} = \tfrac{156}{100} \text{ or } 1\tfrac{56}{100} \text{ or } 1\tfrac{14}{25} \text{ or } \mathbf{1.56}$$

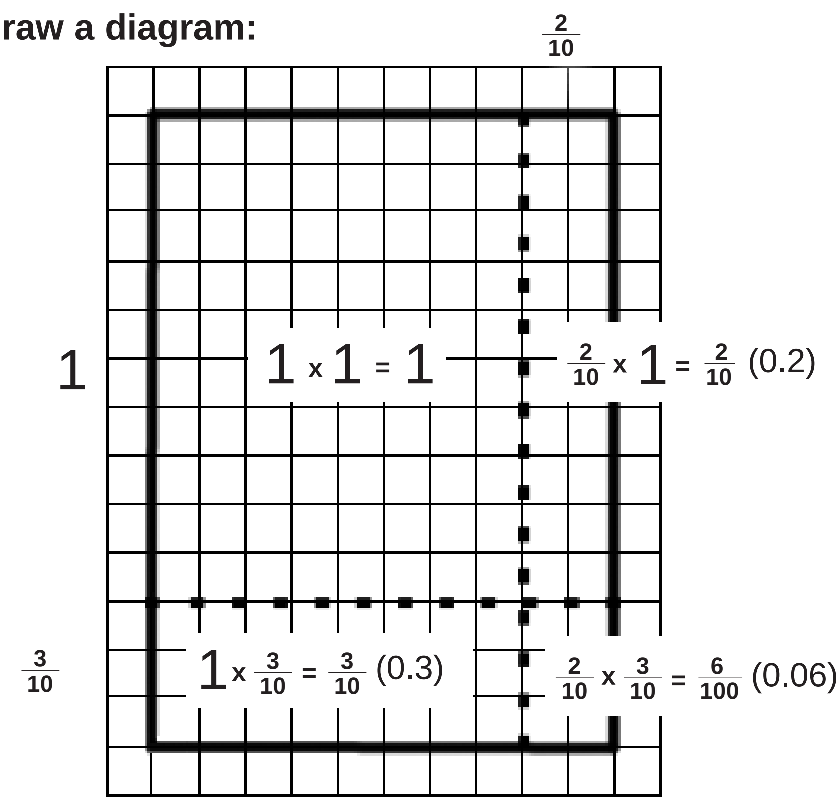
$$\downarrow \quad \downarrow$$

$$1.3 \times 1.2 \longrightarrow \begin{array}{r} 13 \\ 12 \\ \hline 6 \\ 20 \\ 30 \\ 100 \\ \hline 156 \end{array}$$

Complete the question as whole numbers using the distributive principle and then insert the decimal to make the answer make sense.

If 12 x 13 = 156 then 1.2 x 1.3 has to be slightly more than 1. Therefore, the answer is 1.56.

Also draw a diagram:



$$\begin{array}{r} \mathbf{1} \\ \mathbf{0.2} \\ \mathbf{0.3} \\ \mathbf{0.06} \\ \hline \mathbf{1.56} \end{array}$$

It is not important that students are able to replicate this process independently on paper, as attempting this process may create confusion for some students. If, however, the process is attempted a number of times in class as a whole-group activity guided by teacher input, students will gradually begin to learn how to estimate answers and how to predict the size of parts (denominators) before they decide on a final answer.

Example One:

$3\frac{3}{5} \times 4\frac{2}{3}$

The answer should be more than 12 and less than 20.

$\frac{18}{5} \times \frac{12}{3}$

Use the improprer fractional format as it is easier.

$\frac{\cancel{3} \times 6 \times 12}{\cancel{3} \times 5}$

Rewrite in fractional format in order to recognize common factors, or divide by 1 (3/3 os the GCF).

$\frac{72}{5}$ or $14\frac{2}{5}$ or 14.4

Example Two:

$1\frac{3}{10} \times 3\frac{1}{4}$

Answer greater than 3 and less than 6.

1.3×3.25

Use the whole number format and then introduce decimals

$$\begin{array}{r} 3.25 \\ \underline{1.3} \\ 15 \\ 60 \\ 900 \\ 50 \\ 200 \\ \underline{3000} \\ 4.225 \end{array}$$

Use fractional format.

$\frac{13}{10} \times \frac{13}{4} \longrightarrow \frac{169}{40}$ or $4\frac{9}{40}$

Dividing Fractions

Students will learn how to divide fractions far more easily where no rote rules are introduced. Begin instruction by focusing upon understandings that students have already acquired (such the division of a whole number into fractional parts).

Example One:

The basketball coach buys 24 oranges for the upcoming game and cuts the oranges into quarter.
How many pieces will be get from the 24 oranges?
How many pieces will each of the 12 players receive?

$24 \div \frac{1}{4}$

$1 \div \frac{1}{4} = 4$

$2 \div \frac{1}{4} = 8$

$3 \div \frac{1}{4} = 12$

so $24 \div \frac{1}{4}$ must be $24 \times \frac{4}{1} = 96$

(note that **4** and **24** are equivalent ways of writing $\frac{4}{1}$ and $\frac{24}{1}$)

so $\frac{24}{1} \times \frac{4}{1} = \frac{96}{1} = 96$

Next $96 \div 12 = 8$ or $\frac{96}{1} \div \frac{12}{1} = \frac{8}{1} = 8$

Each player will receive eight pieces, or two whole oranges.

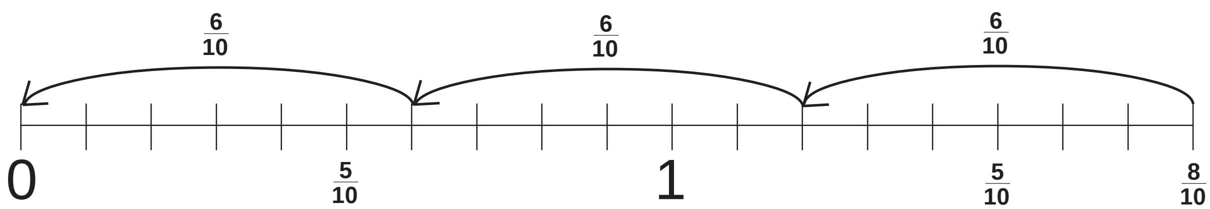
Note that this division process parallels the process of the multiplication of fractions.

The astute student will notice that the two orange answer would be easily arrived at by doing 24 ÷ 12 = 2.

Now ask students to attempt a more difficult problem.

Example Two: [part A]

A tailor has one and eight-tenths metres of cloth, and he needs to cut the cloth into strips of <u>six-tenths</u>. How many strips will the tailor be able to cut?



Division can be interpreted as multiple subtractions.

How many times can $\frac{6}{10}$ be subtracted from $1\frac{8}{10}$?

The answer is 3 .

$$\frac{18}{10} \div \frac{6}{10} \longrightarrow \frac{18 \div 6}{10 \div 10} = \frac{3}{1} = 3$$

Again, this parallels the multiplication process.

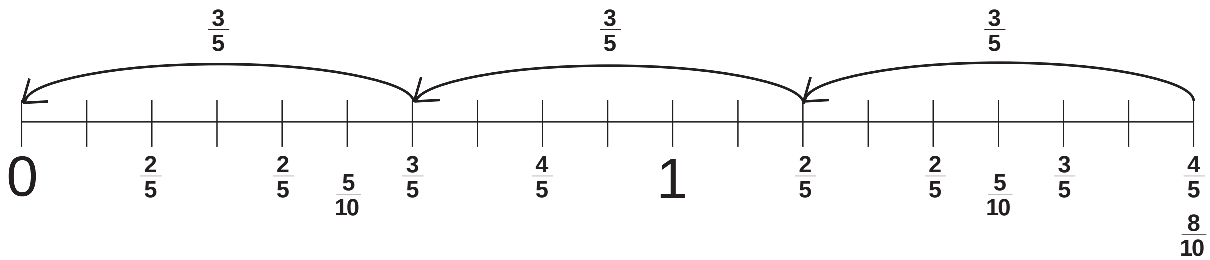
In this example, the denominators are the same, and therefore the process is a simple one.

Ask your students: “What happens if the denominators are different?”

Now try the same questions worded differently in order to include equivalent fractions.

Example Two: [part B]

The tailor has one and eight-tenths metres of cloth,
and he needs to cut the cloth into strips measuring <u>three-fifths</u> of a metre in length.
How many strips will the tailor be able to cut?



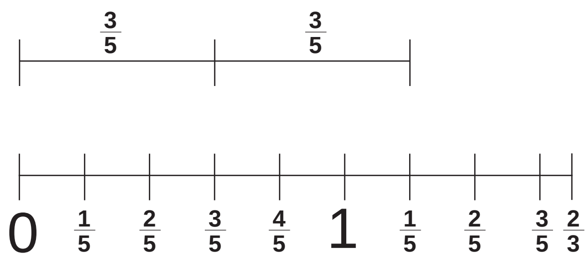
$$\frac{18}{10} \div \frac{3}{5} = \frac{6}{2} = 3$$

Note: This division process parallels the multiplication process in reverse.

We may well ponder: "Why do we not teach this method of dividing fractions on a regular basis in classrooms?" The answer, of course, is that more complex and less reality-based division examples present difficulties.

Example Two: [part C]

The tailor has one and two-thirds metres of cloth, and he needs to cut the cloth into strips measuring <u>three-fifths</u> of a metre. How many strips will the tailor be able to cut from the cloth?



Now make a strip equal to $\frac{3}{5}$ and figure out how many of these $\frac{3}{5}$ strips will fit. The answer is 2 and a partial strip

Try the fraction method (use $\frac{5}{3}$ for $1\frac{2}{3}$).

$\frac{5}{3} \div \frac{3}{5}$ Note how some students who have been taught rules would be included to cancel out.

$\frac{5 \div 3}{3 \div 5}$ This example doesn't work well.

(continued next page)

Example Two: [part C] (continued)

Write in vertical format: $\dfrac{\frac{5}{3}}{\frac{3}{5}}$ Use the "Principle of One" which is what we wish the denominator to be. Therefore, multiply $\frac{3}{5}$ by $\frac{5}{3}$.

This is why invert and multiply works.
It is the "Principle of One" and the reciprocal.

$$\dfrac{\frac{5}{3}}{\frac{3}{5}} \times \boxed{\dfrac{\frac{5}{3}}{\frac{5}{3}}} \text{ (makes a one)} \quad \dfrac{\frac{25}{9}}{\frac{15}{15}} = \dfrac{\frac{25}{9}}{1} = \frac{25}{9} \text{ or } 2\frac{7}{9}$$

An alternate method is to find the LCD of both fractions.

$$\frac{5}{3} \times \frac{5}{5} = \frac{25}{15} \qquad \frac{3}{5} \times \frac{3}{3} = \frac{9}{15}$$

$$\frac{25}{15} \div \frac{9}{15} = \dfrac{\frac{25}{9}}{1} = \frac{25}{9}$$

When teaching the division of fractions using this process you will soon recognize that teaching the application of the "invert and multiply" rule is far faster in the short term. However, the application of the "invert and multiply" rule reduces understanding in the long term. The "invert and multiply" rule has no other application in the mathematical world. Nevertheless, many students continue in vain to try to apply this rule inappropriately. Students often attempt to apply the rules they learn in one context to other, unrelated contexts. Thus, we often encounter students attempting to cross-multiply, invert and multiply, or "cancel out" in quite inappropriate contexts.

Understanding fractions is mostly based upon a thorough understanding of the ***principle of one***, the ***principle of balance***, and the ***principle of equivalence***, combined with the one, unifying principle of: "***Does my answer make sense?***"

Attach a Hundredth Square FORM A Write a Story Problem on the Back		
	Fraction	Picture (Diagram)
	Decimal	
	Percent	Number Line 0 1
	Ratio	

Choose a Fraction > 1 and < 2 FORM B Write a Story Problem on the Back		
	Fraction	Picture (Diagram)
	Decimal	
	Percent	Number Line 0 1 2
	Ratio	

TEACHING NOTES